

Solution of Deny Convolution Equation Restricted to a Half Line via a Random Walk Approach

ABDULHAMID A. ALZAIID

King Saud University, Saudi Arabia,

KA-SING LAU AND C. RADHAKRISHNA RAO*

University of Pittsburgh,

AND

D. N. SHANBHAG*

University of Sheffield, United Kingdom

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A general solution of the Deny convolution equation restricted to a half line is obtained using certain concepts of random walk theory. The equation in question arises in several places in applied probability such as in queueing and storage theories and characterization problems of probability distributions. Some of the important applications are briefly discussed. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let S be such that it equals either $R(= (-\infty, \infty))$ or $R_+(= [0, \infty))$, σ be a σ -finite measure on S such that $\sigma(\{0\}) < 1$ and $H: S \rightarrow R_+$ be a Borel measurable function that is locally integrable (w.r.t. Lebesgue measure), satisfying the integral equation

$$H(x) = \int_S H(x+y) \sigma(dy) \quad \text{for a.a. } [L]x \in S, \quad (1.1)$$

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where a.a. $[L]$ refers to "almost all w.r.t. Lebesgue measure." The integral equation (1.1) when $S = R$ or $S = R_+$ has been extensively studied by many authors. In particular, Deny [8] has identified the general solution to the equation when $S = R$ and Lau and Rao [13] when $S = R_+$. (Indeed, Deny [8] arrives at the solution to the equation with $S = R$ assuming H to be continuous; however, as a straightforward corollary of Deny's theorem, the general solution when H is not necessarily continuous follows as pointed out by Rao and Shanbhag [25].) A special case of Deny's result when σ is a probability measure and H is bounded was established earlier by Choquet and Deny [6], while the Lau-Rao result subsumes various partial results given earlier by Marsaglia and Tubilla [17], Shanbhag [28], Shimizu [30], Ramachandran [19], and several others.

There also exist by now a number of alternative approaches for arriving at the solution to (1.1) either in the case of $S = R$ or $S = R_+$. (See, for example, Ramachandran [20], Davies and Shanbhag [7], Ramachandran and Prakasa Rao [21], Lau and Rao [14, 15], Rao and Shanbhag [24] and Alzaid, Rao, and Shanbhag [3]). Both Deny's theorem and its variant given by Lau and Rao have applications in characterization problems of probability distributions and branches of applied probability such as reliability and renewal theories. For the details concerning applications, the reader is referred to Feller [9, Vol. 2, p. 351], Shimizu [30], Shanbhag [28], Lau and Rao [13], Rao and Shanbhag [25], Rao [22], Alzaid [1], and Rao and Shanbhag [24] and relevant references therein.

Consider now the integral equation

$$H(x) = \int_R H(x+y) \sigma(dy) \quad \text{for a.a. } [L]x \in R_+, \quad (1.2)$$

where σ is a σ -finite measure on R such that $\sigma(\{0\}) < 1$ and H is a nonnegative locally integrable Borel measurable function on R . (It can be shown by choosing H and σ properly that the Lau-Rao [13] equation is a special case of (1.2).) We show that H satisfying (1.2) has the representation

$$H(x) = \int_{(-\infty, 0)} H(x+y) \rho(dy) + \xi(x) e^{\eta x} \quad \text{for a.a. } [L]x \in R_+ \quad (1.3)$$

for some real η , a certain measure ρ concentrated on $(-\infty, 0)$, and a nonnegative periodic function ξ having every nonzero support point of σ as its period. Further the measure ρ arrived at in (1.3) is such that if $\sigma((-\infty, 0)) > 0$, then either both ρ and σ are nonarithmetic or they are arithmetic with the same span.

In view of this result, it is shown in Corollary 1 of Section 2, that Deny's [8] result or its generalized versions given by Lau and Rao [15]

and Prakasa Rao and Ramachandran [21] follow from the Lau-Rao [13] theorem. Since the proof of (1.3) given in this paper is rather simple, the observation just made concerning the alternative proof of Deny's [8] result is of importance, especially in the light of the elementary proof based on exchangeability of the Lau-Rao theorem given recently by Alzaid, Rao, and Shanbhag [3]; the possibility of such an approach to Deny's theorem was pointed out by Alzaid, Rao, and Shanbhag [3]. (The reader may find it instructive to compare the present proof based on the Lau-Rao theorem of Deny's theorem with the proof of the Lau-Rao theorem based on Deny's theorem as given by Rao and Shanbhag [24].)

2. THE MAIN THEOREM

Let us consider Eq. (1.2), and define, relative to σ , the measures σ_{1n} and σ_{2n} on R such that for every Borel set B of R and every integer $n \geq 1$,

$$\sigma_{1n}(B) = \sigma^n(\{(x_1, \dots, x_n) \in R^n : s_m \in R_+, m = 1, \dots, n-1, s_n \in B\})$$

and

$$\begin{aligned} \sigma_{2n}(B) &= \sigma^n(\{(x_1, \dots, x_n) \in R^n : s_m \in (-\infty, 0), m = 1, \dots, n-1, s_n \in B\}) \\ &= \sigma^n(\{(x_1, \dots, x_n) \in R^n : s_m > s_n, m = 1, \dots, n-1, s_n \in B\}), \end{aligned}$$

where $s_m = x_1 + \dots + x_m$, $m \geq 1$ and σ^n is the product measure $\prod_{i=1}^n \sigma_i$ with $\sigma_i = \sigma$ in the notation of Burrill [5]. Following the analogy with concepts in random walk in probability theory, we may refer to the measures ρ and τ defined below respectively as the descending ladder height measure and the (weak) ascending ladder height measure relative to σ ,

$$\rho(\cdot) = \sum_{n=1}^{\infty} \sigma_{1n}((-\infty, 0) \cap \cdot) \quad (2.1)$$

and

$$\tau(\cdot) = \sum_{n=1}^{\infty} \sigma_{2n}(R_+ \cap \cdot). \quad (2.2)$$

It is easy to check that if $\sigma((-\infty, 0)) > 0$, then the closed subgroup of R generated by $\text{supp}[\rho]$ is the same as that generated by $\text{supp}[\sigma]$. Also, if $\sigma((0, \infty)) > 0$, then the closed subgroup of R generated by $\text{supp}[\tau]$ is the same as the one generated by $\text{supp}[\sigma]$. These observations in turn, imply that if $\sigma((-\infty, 0)) > 0$, then either both ρ and σ are nonarithmetic or both are arithmetic with the same span, and if $\sigma((0, \infty)) > 0$, then either both τ

and σ are nonarithmetic or both are arithmetic with the same span. The main result of the paper is the following theorem.

THEOREM. *Let the function H satisfying (1.2) not be a function that is equal to 0 a.e. $[L]$ on (α, ∞) , where $\alpha = \inf(\text{supp}[\sigma])$. Then ρ and τ as defined in (2.1) and (2.2) are Lebesgue–Stieltjes measures. (This is equivalent to the statement that they are both Radon measures and also the statement that they are both regular measures.) Moreover,*

$$H(x) = \int_{(-\infty, 0)} H(x+y) \rho(dy) + \xi(x) e^{\eta x} \quad \text{for a.a. } [L] x \in R_+, \quad (2.3)$$

where ξ is a nonnegative periodic function with $\xi(\cdot) = \xi(\cdot + s)$ for every $s \in \text{supp}[\sigma]$, and η is a real number such that

$$\int_{R_+} e^{\eta x} \tau(dx) = 1. \quad (2.4)$$

(If η satisfying (2.4) does not exist, then we take ξ in (2.3) to be identically zero with η as an arbitrary real number.)

Proof. The case $\sigma((0, \infty)) = 0$ is trivial, since (1.2) implies, in view of the assumptions, that σ is a Lebesgue–Stieltjes measure in this case. We shall therefore assume that $\sigma((0, \infty)) > 0$. The problem remains invariant if H and $\sigma(dy)$ are replaced respectively by the function $H(x) e^{-\delta x}$, $x \in R$, and $e^{\delta y} \sigma(dy)$ for any $\delta \in R$. In that case, there is no loss of generality in assuming that $\sigma((0, \infty)) > 1$ and hence in assuming that

$$0 < H^*(x) \triangleq \int_x^\infty H(y) dy < \infty \quad \text{for all } x \in R$$

following essentially the arguments of Alzaid *et al.* [3]. Since (1.2) is satisfied by H^* , the theorem follows if we prove it by replacing H by H^* , provided in this case it is found additionally that (2.3) has $\xi(x) \exp(\eta x)$, $x \in R_+$ to be decreasing. We can then assume without loss of generality that H itself is a positive decreasing continuous function. Now (1.2) gives us inductively, on using Fubini's theorem,

$$H(x) = \int_{(-\infty, 0)} H(x+y) \left(\sum_{n=1}^k \sigma_{1n} \right) (dy) + \int_{R_+} H(x+y) \sigma_{1k}(dy) \quad (2.5)$$

$$k = 1, 2, \dots; \quad x \in R_+,$$

because $\int_{R_+} H(x+y) \sigma_{1k}(dy) = \int_R H(x+y) \sigma_{1k+1}(dy)$, $k = 1, 2, \dots$, $x \in R_+$. This, in turn, implies that for each $x \in R_+$, the sequence of the latter

integrals on the r.h.s. of (2.5) is a decreasing (nonnegative real) sequence and hence has a limit. Consequently, we can write

$$H(x) = \int_{(-\infty, 0)} H(x+y) \rho(dy) + \hat{H}(x), \quad x \in R_+, \quad (2.6)$$

where

$$\hat{H}(x) = \lim_{k \rightarrow \infty} \int_{R_+} H(x+y) \sigma_{1k}(dy); \quad (2.7)$$

it is obvious that the function $\hat{H}(x)$, $x \in R_+$ appearing here is nonnegative and, like H , is decreasing continuous.

Since H is assumed to be positive continuous and the integral appearing on the r.h.s. of (2.6) has to be finite, it follows immediately that ρ is a Lebesgue–Stieltjes measure. (Indeed (2.6) implies, in view of the decreasing nature of H , that $\rho(R) \leq 1$.) The fact that τ is a Lebesgue–Stieltjes measure is obvious in the case of $\sigma((-\infty, 0)) = 0$, in view of the property of H , since

$$\tau = \sigma \quad \text{and} \quad \int_{R_+} H(y) \sigma(dy) < \infty.$$

Furthermore, if $\sigma((-\infty, 0)) > 0$, then given any $s_0 \in (-\infty, 0) \cap \text{supp}[\sigma]$ and $y \in R_+$, we have

$$\begin{aligned} & \tau\left(\left(-\infty, y + \frac{ns_0}{2}\right)\right) \sigma\left(\left(\frac{3s_0}{2}, \frac{s_0}{2}\right)\right) \\ & \leq \tau\left(\left(-\infty, y + \frac{(n+1)s_0}{2}\right)\right) + \rho\left(\left(\frac{3s_0}{2}, 0\right)\right), \quad n = 0, 1, 2, \dots, \end{aligned} \quad (2.8)$$

which implies that

$$\tau\left(\left(-\infty, y + \frac{ns_0}{2}\right)\right) < \infty \quad \text{if} \quad \tau\left(\left(-\infty, y + \frac{(n+1)s_0}{2}\right)\right) < \infty.$$

Since $\tau((-\infty, 0)) = 0$, we have then by reverse induction that $\tau((-\infty, y)) < \infty$ for each y . This implies that τ is a Lebesgue–Stieltjes measure. (For obtaining (2.8), the identity given in brackets immediately after the definition of σ_{2n} may be used.) This establishes the first part of the theorem.

Define now, for any Borel set $A(\subset R)$, σ_{1n}^A to be the measure σ_{1n} with

$s_1 \in R_+$ in its definition replaced by $s_1 \in R_+ \cap A$. Clearly then for every $x \in R_+$ and Borel set A the sequence

$$\left\{ \int_{R_+} H(x+y) \sigma_{1n}^A(dy) : n = 1, 2, \dots \right\}$$

is a decreasing sequence of nonnegative real numbers and hence has a limit. Consequently, we have for each $c \in R_+$,

$$\begin{aligned} \hat{H}(x) &= \lim_{n \rightarrow \infty} \int_{R_+} H(x+y) \sigma_{1n}^{[0, c]}(dy) \\ &\quad + \lim_{n \rightarrow \infty} \int_{R_+} H(x+y) \sigma_{1n}^{(c, \infty)}(dy), \quad x \in R_+. \end{aligned} \quad (2.9)$$

Since for every $x, c \in R_+$ and $n = 1, 2, \dots$

$$\int_{R_+} H(x+y) \sigma_{1n}^{(c, \infty)}(dy) \leq \int_{(c, \infty)} H(x+y) \sigma(dy)$$

it follows that the latter limit on the r.h.s. on (2.9) tends to zero as $c \rightarrow \infty$. We can therefore write

$$\hat{H}(x) = \lim_{c \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{R_+} H(x+y) \sigma_{1n}^{[0, c]}(dy), \quad x \in R_+. \quad (2.10)$$

If we now define $\sigma_{2n}^{(c)}$ by the expression for σ_{2n} given under brackets immediately after its definition with $s_1 > s_n$ replaced by $c \geq s_1 > s_n$, then for each $c \in R_+$ and $n = 1, 2, \dots$

$$\begin{aligned} &\int_{R_+} H(x+y) \sigma_{1n}^{[0, c]}(dy) \\ &= \sum_{m=1}^n \int_{[0, c]} \int_{R_+} H(x+y+z) \sigma_{1, n-m}(dz) \sigma_{2m}^{(c)}(dy), \quad x \in R_+, \end{aligned} \quad (2.11)$$

where σ_{10} is the probability measure concentrated on $\{0\}$, because

$$R^n = \bigcup_{m=1}^n \{(x_1, \dots, x_n) : s_r > s_m, r = 1, 2, \dots, m-1; s_r \geq s_m, r = m, \dots, n\}.$$

Since τ is a Lebesgue-Stieltjes measure, $\tau^{(c)}(\cdot) \triangleq \sum_{n=1}^{\infty} \sigma_{2n}^{(c)}(R_+ \cap \cdot)$ is increasing in c with limit τ as $c \rightarrow \infty$, and

$$0 \leq \int_{R_+} H(x+y+z) \sigma_{1n}(dz) \leq H(x), \quad x, y \in R_+, n = 1, 2, \dots,$$

we get from (2.10) and (2.11), using the Lebesgue dominated convergence theorem in particular, that

$$\begin{aligned}\hat{H}(x) &= \lim_{c \rightarrow \infty} \int_{[0, c]} \hat{H}(x+y) \tau^{(c)}(dy) \\ &= \int_{R_+} \hat{H}(x+y) \tau(dy), \quad x \in R_+.\end{aligned}\quad (2.12)$$

Then (2.6) (together with the note concerning \hat{H} following it) and (2.12) imply the second part of the theorem, in view of the Lau–Rao [13] theorem, for which an elementary proof exists as shown by Alzaid *et al.* [3].

Remark 1. The above theorem remains valid with trivial changes in its proof even when the local integrability of H on R is replaced by that on (α, ∞) , where α is as defined in the statement of the theorem.

Remark 2. The proof of the above theorem simplifies considerably if (1.2) is replaced by (1.1) with $S = R$. In that case, essentially by symmetry, the fact that ρ is a Lebesgue–Stieltjes measure implies that τ is a Lebesgue–Stieltjes measure. Also, the validity of the identity in (2.6) for all $x \in R$ implies that

$$\int_{(-\infty, 0)} H(x+y) \rho(dy) < \infty \quad \text{for all } x \in R.$$

The derivation of (2.6) from (1.2) remains valid also for H increasing continuous provided we do not require any longer \hat{H} to be decreasing. Using this fact and appealing to the essential symmetry, we have

$$\int_{R_+} H(x+y) \tau(dy) < \infty \quad \text{for all } x \in R.$$

In view of these observations, it follows that, in the present case, the theorem follows even when the portion of the proof following the observations that ρ is a Lebesgue–Stieltjes measure until the identity (2.10) is deleted, provided $[0, c]$, $\sigma_{2n}^{(c)}$, and $\sigma_{1n}^{[0, c]}$ in (2.11) are replaced respectively by R_+ , σ_{2n} , and σ_{1n} and the portions not relevant are deleted.

COROLLARY 1 (Deny [8] and Lau and Rao [15]). *If H satisfying (1.1) with $S = R$ is not a function that is equal to zero a.e. $[L]$, then it has the representation*

$$H(x) = \xi_1(x) \exp(\eta_1 x) + \xi_2(x) \exp(\eta_2 x) \quad \text{for a.a. } [L] x \in R,$$

where ξ_1 and ξ_2 are periodic functions with $\xi_i(\cdot) = \xi_i(\cdot + s)$, $i = 1, 2$, for each point $s \in \text{supp}[\sigma]$ and η_i such that

$$\int_R \exp(\eta_i x) \sigma(dx) = 1, \quad i = 1, 2.$$

(For the uniqueness of the representation but for the ordering of the terms, one may assume for example that $\xi_2 \equiv 0$ if $\eta_1 = \eta_2$; Rao and Shanbhag [24] have implicitly assumed this to be so in their proof of the Lau–Rao [15] result based on Deny's theorem.)

Proof. As in the proof of the main theorem, it is clear that there is no loss of generality in assuming H to be continuous and decreasing. In such a case, we get

$$H(x) = \int_{(-\infty, 0)} H(x+y) \rho(dy) + \xi(x) \exp(\eta x) \quad \text{for all } x \in R \quad (2.13)$$

with ξ as a nonnegative continuous periodic function on R with every non-zero point of $\text{supp}[\sigma]$ as its period and η as a real number. Clearly η in (2.13) satisfies

$$\int_R \exp(\eta x) \sigma(dx) = 1$$

if $\xi \not\equiv 0$. Define now c^* to be equal to zero if $\xi \equiv 0$ and to be the supremum of the set C defined as the set of all nonnegative c 's for which $H_c(x) \triangleq H(x) - c\xi(x) \exp(\eta x) \geq 0$ for all $x \in R$ and

$$H_c(x) = \int_{(-\infty, 0)} H_c(x+y) \rho(dy) + d\xi(x) \exp(\eta x), \quad x \in R$$

with $d \geq 0$. Clearly C is compact by the dominated convergence theorem (on noting in particular that $0 \leq c, d \leq \exp(-\eta x_0) H(x_0)/\xi(x_0) (< \infty)$ when $\xi(x_0) \neq 0$) and hence $c^* \in C$; also

$$H_{c^*}(x) = \int_{(-\infty, 0)} H_{c^*}(x+y) \rho(dy), \quad x \in R. \quad (2.14)$$

In view of the Lau–Rao [13] theorem (which is now an obvious corollary of the main theorem) and the definition of H_{c^*} , the asserted representation for H follows.

Remark 3. If σ is any Lebesgue–Stieltjes measure on R (not necessarily associated with an integral equation), then, using the arguments employed

in the proof of the theorem to establish τ to be a Lebesgue–Stieltjes measure, we have that both ρ and τ as defined in (2.1) and (2.2) respectively are Lebesgue–Stieltjes measures when at least one of them is given to be so. In that case it is easily seen that

$$\sigma(B) + \tau^* \rho(B) = \begin{cases} \rho(B) & \text{if } B \text{ is a Borel set } \subset (-\infty, 0) \\ \tau(B) & \text{if } B \text{ is a Borel set } \subset R_+ \end{cases}, \quad (2.15a)$$

which implies that

$$\sigma + \tau^* \rho = \rho + \tau. \quad (2.15b)$$

(To see (2.15a), it is sufficient to observe that $\rho(B)$ is given by the l.h.s. of the identity for each Borel subset B of $(-\infty, 0)$; the second part of the identity then follows essentially by symmetry.)

Equation (2.15) implies that for every relatively compact Borel subset B or R

$$\sigma(B) = \rho(B) + \tau(B) - \tau^* \rho(B), \quad (2.16)$$

which may be viewed as the Wiener–Hopf factorization of σ . In the proof of the theorem, after observing that ρ and τ are Lebesgue–Stieltjes measures, one could have obviously appealed to either (2.15b) or (2.16) to arrive at the result, given the condition

$$\int_{R_+} H(x+y) \tau(dy) < \infty \quad \text{for all } x \in R_+,$$

since this yields

$$\begin{aligned} \hat{H}(x) &= H(x) - \int_{(-\infty, 0)} H(x+y) \rho(dy) \\ &= \int_R H(x+y) \sigma(dy) - \int_R H(x+y) \rho(dy) \\ &= \int_R \left[H(x+y) - \int_R H(x+y+z) \rho(dz) \right] \tau(dy) \\ &= \int_{R_+} \hat{H}(x+y) \tau(dy), \quad x \in R_+. \end{aligned}$$

In the case of (1.1) with $S = R$, we have

$$\int_{R_+} H(x+y) \tau(dy) < \infty, \quad x \in R$$

as observed in Remark 2, and hence the present approach remains valid. This provides us with a further proof of the result of Corollary 1.

Remark 4. In view of Remark 3, we may raise the question as to whether there exists, under the hypothesis of the theorem, a situation in which for some Borel $B \subset R_+$ with positive Lebesgue measure

$$\int_{R_+} H(x+y) \tau(dy) = \infty \quad \text{for all } x \in B$$

so that the argument based on the Wiener–Hopf factorization as it stands does not remain valid. The answer to this question is in the affirmative as is shown in the following example.

EXAMPLE 1. Let σ be the probability measure concentrated on $\{-1, 0, 1, 2, \dots\}$ such that its moment generating function is given by

$$M(t) = 1 + \alpha e^{-t}(1 - e^t)^\beta, \quad t \leq 0,$$

where α and β are fixed positive numbers such that $1 < \beta < 2$ and $0 < \alpha\beta \leq 1$. (This moment generating function, but for a location change, was considered earlier by Seneta [26] in connection with a certain problem in branching processes.) Let H be such that

$$H(x) = \begin{cases} [x+1] & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $[x+1]$ is the integer part of $x+1$. Hence it follows that H satisfies the hypothesis of the theorem and

$$H(x) = \int_R H(x+y) \sigma(dy), \quad x \in R_+.$$

In this case ρ is the probability measure concentrated on $\{-1\}$ and, in view of the Wiener–Hopf factorization of σ , τ is the probability measure with the moment generating function

$$M^*(t) = 1 - \alpha(1 - e^t)^{\beta-1}, \quad t \leq 0.$$

The expression for $M^*(t)$ implies that τ in question has an infinite mean and hence we have here

$$\int_{R_+} H(x+y) \tau(dy) = \infty \quad \text{for each } x \in R_+.$$

Thus we have a simple counterexample supporting the claim made. From this example we can obviously produce examples with H satisfying an additional condition of being positive continuous and decreasing. (It is also worth pointing out here that this example illustrates the validity of the statement of Feller [9, p. 380 above Theorem 2] concerning ladder height means in a random walk that is induced by variables with zero mean; in the present case, we have the mean of σ to be zero, the mean of the descending ladder height to be finite, and the mean of the ascending ladder height to be infinite.)

Remark 5. In (2.3), we can choose ξ to be a constant if σ is non-arithmetic and as a periodic function with period λ if σ is arithmetic with span λ .

Remark 6. For the σ appearing in the theorem, the Wiener–Hopf factorization given by (2.15) implies that for every real θ and x, y such that $0 \leq x < y < \infty$,

$$\int_{[x, y]} \exp(\theta z) \tau(dz) \int_{[-x, 0]} \exp(\theta z) \rho(dz) \leq \int_{[0, y]} \exp(\theta z) \tau(dz)$$

yielding that

$$\tau^*(\theta) \triangleq \int_{R_+} \exp(\theta z) \tau(dz) < \infty$$

whenever

$$\rho^*(\theta) \triangleq \int_{(-\infty, 0)} \exp(\theta z) \rho(dz) > 1.$$

(This is essentially the argument used in Alzaid *et al.* [3] for showing a certain integral $\int_0^\infty f(y) dy$ to be finite.)

By symmetry, we have also $\rho^*(\theta) < \infty$ whenever $\tau^*(\theta) > 1$. The Wiener–Hopf factorization of σ also gives

$$\sigma^*(\theta) + \tau^*(\theta) \rho^*(\theta) = \tau^*(\theta) + \rho^*(\theta), \quad \theta \in R, \quad (2.17)$$

where

$$\sigma^*(\theta) = \int_R \exp(\theta x) \sigma(dx).$$

Then (2.17) implies the following. If we assume that $\tau^*(\theta) > 1$ and $\rho^*(\theta) > 1$ (and hence that $1 < \tau^*(\theta) < \infty$ and $1 < \rho^*(\theta) < \infty$), then we get

that $1 - \sigma^*(\theta) (= (1 - \tau^*(\theta))(1 - \rho^*(\theta))) > 0$ and hence $\sigma^*(\theta) < 1$. However, the definitions of ρ^* and τ^* imply that $\rho^*(\theta)$ and $\tau^*(\theta)$ denote respectively $\rho((-\infty, 0))$ and $\tau(R_+)$ corresponding to the case with $e^{\theta x}\sigma(dx)$ in place of $\sigma(dx)$. If $\sigma^*(\theta) < 1$, we have obviously the measure relative to $e^{\theta x}\sigma(dx)$ (i.e., the measure for which each Borel set B receives the value $\int_B e^{\theta x}\sigma(dx)$) to be bounded by 1 and hence the ladder height measure interpretations of τ and ρ yields that $\rho^*(\theta) < 1$ and $\tau^*(\theta) < 1$. Consequently it follows that for each θ at least one of $\tau^*(\theta)$ and $\rho^*(\theta)$ should be less than or equal to 1. (Otherwise, we arrive at a contradiction.) Now, if θ_0 is such that $\tau^*(\theta_0) = 1$ (which implies that $\tau^*(\theta) > 1$ for each $\theta > \theta_0$ since $\sigma(\{0\}) < 1$) or $\rho^*(\theta_0) = 1$ (which implies that $\rho^*(\theta) > 1$ for each $\theta < \theta_0$), then we can conclude, using the monotone convergence theorem, that $\tau^*(\theta_0) = 1$ and $\rho^*(\theta_0) \leq 1$ or $\rho^*(\theta_0) = 1$ and $\tau^*(\theta_0) \leq 1$ and hence that $\sigma^*(\theta_0) = 1$; this, in turn, yields that if (2.4) holds, then $\sigma^*(\eta) = 1$.

Remark 7. From what is given in Remark 6, we have that the measure σ in the theorem satisfies either $\sigma((-\infty, 0)) \leq 1$ or $\sigma(R_+) \leq 1$.

COROLLARY 2. *Let H be as in the theorem and for each $x \in R_+$, let $\rho^{(x)}$ denote the ρ measure on R with an alteration that the s_r 's involved in its definition are replaced by $s_r + x$. Then, for each $x \in R_+$, $\rho^{(x)}$ is a Lebesgue–Stieltjes measure and, given that*

$$H(x) \quad \text{and} \quad \int_{(-\infty, 0)} H(x + y) \rho(dy)$$

are right continuous when restricted to R_+ , we have for each $x \in R_+$,

$$H(x) = \int_{(-\infty, 0)} H(y) \rho^{(x)}(dy) + \xi(x) \int_{[-x, 0]} e^{\eta(x+y)} \left(\sum_{n=0}^{\infty} \rho^{*n} \right) (dy) \quad (2.18)$$

*for some nonnegative real periodic function ξ such that it is a constant if σ is nonarithmetic and a function with period λ if σ is arithmetic with span λ , and η is as defined in the theorem. (If η satisfying (2.4) does not exist, we take η to be any arbitrary real number with $\xi \equiv 0$; also, we define, as usual ρ^{*n} for $n \geq 1$ to be the n -fold convolution of ρ with itself and ρ^{*0} to be the probability measure that is degenerate at zero.) Moreover, if f is any Borel measurable function on $(-\infty, 0)$ such that $\int_{(-\infty, 0)} |f(y)| \rho^{(x)}(dy) < \infty$ for each $x \in R_+$ and ξ^* is any Borel measurable function of the form of ξ defined above with a modification that it is not necessarily nonnegative, then the function \hat{H} given by*

$$\begin{aligned} \hat{H}(x) &= \int_{(-\infty, 0)} f(y) \rho^{(x)}(dy) + \xi^*(x) \int_{[-x, 0]} e^{\eta(x+y)} \left(\sum_{n=0}^{\infty} \rho^{*n} \right) (dy), \quad x \in R_+ \\ &= f(x), \quad x \in (-\infty, 0) \end{aligned}$$

satisfies (1.2) (even with “a.a. $[L]$ ” replaced by “all”).

Proof. We have that ρ is a Lebesgue–Stieltjes measure concentrated on $(-\infty, 0)$. Clearly, for each $x \in R_+$ and Borel set B

$$\rho^{(x)}(B+x) = \sum_{n=1}^{\infty} \rho_n^{(x)}((-\infty, -x) \cap B), \tag{2.19}$$

where $\rho_1^{(x)} = \rho$ and for each $n \geq 2$, $\rho_n^{(x)}$ is the convolution of measures $\rho_{n-1}^{(x)}([-x, 0) \cap \cdot)$ and ρ (and hence is a well-defined Lebesgue–Stieltjes measure concentrated on $(-\infty, 0)$). For any bounded interval $[\alpha, \beta]$, we have

$$\rho^{(x)}([\alpha, \beta] + x) \leq \sum_{n=1}^{\infty} \rho^{*n}([\alpha, \beta]) < \infty$$

and hence it follows that $\rho^{(x)}$ is a Lebesgue–Stieltjes measure. Now, if $H(x)$ and $\int_{(-\infty, 0)} H(x+y) \rho(dy)$ are right-continuous on R_+ , then (2.3) is valid with “a.a. $[L]$ ” replaced by “all” and ξ as defined in (2.18). From this we get for each positive integer k and $x \in R_+$.

$$\begin{aligned} H(x) = & \int_{(-\infty, -x)} H(x+y) \left(\sum_{n=1}^k \rho_n^{(x)} \right) (dy) + \int_{[-x, 0)} H(x+y) \rho^{*k}(dy) \\ & + \xi(x) \int_{[-x, 0]} e^{\eta(x+y)} \left(\sum_{n=0}^{k-1} \rho^{*n} \right) (dy) \end{aligned} \tag{2.20}$$

on noting that the restrictions of ρ^{*n} and $\rho_n^{(x)}$ to $[-x, 0)$ are identical for each $n \geq 1$. On taking, under the right continuity assumption of $H(x)$ on R_+ (or indeed just at the point 0), the limit as $k \rightarrow \infty$ in (2.20), we arrive at the identity (2.18). (Note that the assumption yields that as $k \rightarrow \infty$, $\int_{[-x, 0)} H(x+y) \rho^{*k}(dy) \rightarrow 0$ for each $x \in R_+$.) It is easy to verify that \hat{H} of the corollary satisfies (1.2) (even with “a.a. $[L]$ ” replaced by “all”) on using the observation in Remark 6 that (2.4) implies $\sigma^*(\eta) = 1$ (and $\rho^*(\eta) \leq 1$) and the counterpart of Feller [9, Eq. (3.7b) of Chapter XII] relative to the random walk $\{-S_n\}$ in place of $\{S_n\}$.

Remark 8. If H is monotonic and right-continuous or, in the case of α finite, if the restriction of H to $[\alpha, \infty)$ is locally bounded (i.e., bounded on each bounded subset of $[\alpha, \infty)$) and right-continuous, then the assumption of Corollary 2 that $H(x)$ and $\int_{(-\infty, 0)} H(x+y) \rho(dy)$ are right-continuous when restricted to R_+ is met.

COROLLARY 3. *Let σ and H be as in the theorem with $\alpha > -\infty$ and let $\rho^{(x)}$ for each $x \in R_+$ be as defined in Corollary 2. Assume that the restriction of H to $[\alpha, \infty)$ is locally bounded and right-continuous. Define $\Theta = \{\theta: \sigma^*(\theta) = 1\}$ with σ^* as defined in Remark 6. Then Θ is nonempty*

(without having more than two members), and if $\sigma((-\infty, 0)) > 0$ and θ_0 is the only member in Θ or is the smaller of the two members in Θ , we have

$$\mu_{\theta_0, \sigma} \triangleq \int_{[\alpha, \infty)} x e^{\theta_0 x} \sigma(dx) \leq 0.$$

Moreover, in this case, we have the assertions of Corollary 2 as valid with

- (i) $\xi \equiv 0$ if Θ is a singleton and $\mu_{\theta_0, \sigma} < 0$,
- (ii) $\int_{[-x, 0]} e^{\eta(x+y)} (\sum_{n=0}^{\infty} \rho^{*n})(dy)$ replaced by $x e^{\theta_0 x} - \int_{(-\infty, 0)} y e^{\theta_0 y} \rho^{(x)}(dy)$ if Θ is a singleton and $\mu_{\theta_0, \sigma} = 0$,
- (iii) $\int_{[-x, 0]} e^{\eta(x+y)} (\sum_{n=0}^{\infty} \rho^{*n})(dy)$ replaced by $e^{\theta_1 x} - \int_{(-\infty, 0)} e^{\theta_1 y} \rho^{(x)}(dy)$ with $\theta_1 \in \Theta$ and $\theta_1 > \theta_0$ if Θ is a doubleton.

Proof. If $\sigma((-\infty, 0)) > 0$, we have obviously in view of the fact that $\alpha > -\infty$ some real number δ such that $\rho^*(\delta) = 1$ and if $\sigma((-\infty, 0)) = 0$, we have clearly $\xi \neq 0$ and, hence from the theorem, η exists such that $\tau^*(\eta) = 1$, where ρ^* and τ^* are as defined in Remark 6. From what is mentioned in Remark 6, we have $\sigma^*(\delta) = 1$ if $\rho^*(\delta) = 1$ and $\sigma^*(\eta) = 1$ if $\tau^*(\eta) = 1$. Consequently it follows that Θ is nonempty. Clearly the assumption $\sigma(\{0\}) < 1$ implies, in view of Remark 6, that $\delta \leq \eta$ whenever $\rho^*(\delta) = 1$ and $\tau^*(\eta) = 1$. From the Wiener-Hopf factorization concerning probability measures, it is clear that for any λ , $\sigma^*(\lambda) = 1$ if and only if either $\rho^*(\lambda) = 1$ or $\tau^*(\lambda) = 1$. It is therefore clear that Θ can at most have two points. (This latter fact also follows directly from properties of moment generating functions of probability measures.) From Feller [9, Theorem 2, p. 380 and the remark following Theorem 2, p. 396], it follows that if $\rho^*(\theta_0) = 1$, then $\mu_{\theta_0, \sigma} \leq 0$. If θ_0 is as defined in the statement of the corollary and $\sigma((-\infty, 0)) > 0$, then, from the fact that $\rho^*(\delta) = 1$ and $\tau^*(\eta) = 1$ imply that $\delta \leq \eta$, we have $\rho^*(\theta_0) = 1$ and hence $\mu_{\theta_0, \sigma} \leq 0$. We shall now establish assertions (i), (ii), and (iii). Clearly, assertion (i) is obvious from Corollary 2 since the remark in Feller [9, following Theorem 2, p. 396] implies $\tau^*(\theta_0) < 1$ if $\mu_{\theta_0, \sigma} < 0$ and, since $\Theta = \{\theta_0\}$, there is no η such that $\tau^*(\eta) = 1$. On the other hand, if we have the situation either as in assertion (ii) or assertion (iii), we have an η such that $\tau^*(\eta) = 1$. In the case of $\mu_{\theta_0, \sigma} = 0$, Feller's Theorem 2 on page 380 implies the existence of $\eta = \theta_0$ and, in the case of Θ containing two points, clearly we have $\eta = \theta_1 > \theta_0$ with $\theta_1 \in \Theta$ in view of what we have observed earlier. Let ξ be as in Corollary 2. We shall consider here this to be a function defined on R . Clearly the restriction of ξ to $[\alpha, \infty)$ is right-continuous and locally bounded. Define now in the case of assertion (ii)

$$H(x) = \begin{cases} \xi(x)(x - \alpha) e^{\theta_0 x} & \text{if } x \geq \alpha \\ 0 & \text{otherwise} \end{cases}$$

and in the case of assertion (iii)

$$H(x) = \begin{cases} \xi(x) e^{\theta_1 x} & \text{if } x \geq \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Observe that in both cases H satisfies (1.2) and we get

$$H(x) = \int_{(-\infty, 0)} H(x + y) \rho(dy) + c\xi(x) e^{\eta x}, \quad x \in R_+$$

for some positive constant c (which need not be the same in the two cases). In both cases, we have H satisfying the conditions required to arrive at (2.18) of Corollary 2 and hence the equation in question is valid with $c\xi$ replacing ξ . Since $c^{-1}\xi$ is of the form of ξ , the assertions (ii) and (iii) easily follow.

COROLLARY 4. *Let $\{(v_n, w_n): n = 0, 1, 2, \dots\}$ be a sequence of vectors of nonnegative real components such that at least one $v_n \neq 0$ and $w_0 > 0$ and $\lambda > 0$. Further, let k and γ be positive integers such that the largest common divisor of k and those n for which $w_n > 0$ be γ . Then the sequence $\{(v_n, w_n)\}$ as defined satisfies the recurrence equations*

$$v_{n+k} = \sum_{m=0}^{\infty} w_m v_{m+n}, \quad n = 0, 1, \dots \tag{2.21}$$

and for some integer k_1 that is an integer multiple of γ such that $k_1 > k$,

$$\lambda v_{n+k} = v_{n+k_1}, \quad n = 0, 1, \dots, k + \gamma - 1 \tag{2.22}$$

only if

$$v_n = q_n \lambda^{n/(k_1-k)}, \quad n = 0, 1, \dots$$

for some nonnegative periodic sequence $\{q_n\}$ with period γ .

Proof. It is sufficient if the result is established for $\gamma = 1$. Defining H and σ such that

$$H(x) = \begin{cases} v_{[x]+k}, & x \geq -k \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\sigma(\{n\}) = w_{n+k}, \quad n \geq -k, \sigma(\{-k, -k+1, \dots\}^c) = 0,$$

where $[x]$ denotes the integer part of x , we see that the conditions for the validity of (2.3) with "a.a. $[L]$ " replaced by "all" are met. The equation in

this case implies, in view of (2.22) and the fact that $\sigma(\{-k\}) > 0$, that $H(x + k_1 - k) = \lambda H(x)$ for all $x \geq -k$. Since we can find an integer r such that $r(k_1 - k) \geq k$, we have subsequently

$$H(x) = \int_{[-k, \infty)} H(x + r(k_1 - k) + y) \lambda^{-r} \sigma(dy), \quad x \in R_+ \quad (2.23)$$

with $r(k_1 - k) - k \geq 0$. There is no loss of generality in assuming that $\sigma(\{0\}) > 0$ and hence that the measure σ^* defined such that

$$\sigma^*(B) = \lambda^{-r} \sigma(B - r(k_1 - k)) \quad \text{for every Borel set } B$$

is arithmetic with unit span. Consequently, the Lau-Rao theorem (which is now a corollary of our theorem) implies, in view of (2.23) and the periodicity of $H(x) \lambda^{-x/(k_1 - k)}$, $x \geq -k$, the required result.

Remark 9. It is also possible to prove the result of the corollary by using the Perron-Frobenius theorem given in Seneta [27, pp. 1-2]. For the details, the reader is referred to Alzaid [1].

Remark 10. If H is as in the theorem and additionally it is locally bounded and right-continuous and α defined in the theorem is finite and $\sigma(\{\alpha\}) > 0$, then, for some β , $\beta^* > |\alpha|$ with these as integer multiples of the span of σ in case σ is arithmetic, and $\lambda > 0$,

$$\lambda H(x + |\alpha|) = H(x + \beta) \quad \text{for all } x \in [0, \beta^*)$$

if and only if $H(x) = \xi(x) \exp(\eta x)$, $x \in [\alpha, \infty)$ with ξ as a periodic function with every nonzero support point of σ to be its period and η such that $\int_{[\alpha, \infty)} \exp(\eta x) \sigma(dx) = 1$ and $\eta = (\beta - |\alpha|)^{-1} \log \lambda$. This result is a version of the above Corollary 4 in the case of arithmetic σ , and it follows via a modified version of the proof of the corollary on using the result of Corollary 2 when $H(\cdot)$ is replaced by $H(x_0 + \cdot)$ with $x_0 \in R_+$ in the case of nonarithmetic σ . Whether or not the result of the present remark is valid without the condition $\sigma(\{\alpha\}) > 0$ is an interesting question to which we have not found any answer as yet.

Remark 11. Now, if we have k to be a positive integer and $\{(v_n, w_n): n = 0, 1, 2, \dots\}$ to be a sequence of vectors with nonnegative real components such that $w_0 > 0$, $v_n \neq 0$ for some n , and the largest common divisor of k and those n for which $w_n > 0$ equals 1, then it follows that the result of Corollary 3 identifies the solution to the system of equations

$$v_{n+k} = \sum_{m=0}^{\infty} w_m v_{m+n}, \quad n = 0, 1, 2, \dots \quad (2.24)$$

given the sequence $\{w_n: n=0, 1, 2, \dots\}$ and the values of v_0, v_1, \dots, v_{k-1} . Indeed, if we define $D = \{b: b > 0, b^k = \sum_{n=0}^{\infty} b^n w_n\}$ and $\{f_{i,r}: i \geq k, r=0, 1, \dots, k-1\}$ to be the sequence of absorption measures corresponding to the nonnegative matrix T (in the sense of Seneta [27]) with state space $\{0, 1, 2, \dots\}$ and the (i, j) th element as

$$T_{ij} = \begin{cases} \delta_{ij} & \text{if } i, j = 0, 1, \dots, k-1 \\ w_{j-i+k} & \text{if } i = k, k+1, \dots \text{ and } j \geq i-k \\ 0 & \text{otherwise,} \end{cases}$$

where δ_{ij} is the Kronecker delta, then Corollary 3 yields that the system of equations is satisfied if and only if D is nonempty, $f_{n,0}, \dots, f_{n,k-1}$ are finite for each $n \geq k$, and any one of the following conditions holds:

(i) D has only one point, $\sum_{n=0}^{\infty} (n-k) b^n w_n < 0$ for $b \in D$, and

$$v_n = f_{n,0} v_0 + \dots + f_{n,k-1} v_{k-1}, \quad n = k, k+1, \dots$$

(ii) D has only one point, $\sum_{n=0}^{\infty} (n-k) b^n w_n = 0$ for $b \in D$ and for some $c \geq 0$,

$$v_n = f_{n,0}(v_0 - c \cdot 0 \cdot b^0) + \dots + f_{n,k-1}(v_{k-1} - c(k-1) b^{k-1}) + cnb^n, \\ n = k, k+1, \dots,$$

with $b \in D$.

(iii) D contains two points and for some $c \geq 0$

$$v_n = f_{n,0}(v_0 - cb^0) + \dots + f_{n,k-1}(v_{k-1} - cb^{k-1}) + cb^n, \quad n = k, k+1, \dots,$$

with b as the larger of the two members of D .

A direct and substantially simpler proof of this last result without involving the Wiener-Hopf factorization can obviously be given (see, for example, Alzaid [1]).

3. SOME COMMENTS ON PREVIOUS RESULTS

We shall now discuss in the light of our findings some existing results and suggest possible extensions.

3.1. Arnold [4]

Let $Y_{1,n} \leq \dots \leq Y_{n,n}$ denote the n ordered observations in a random sample of size n from a nondegenerate distribution F concentrated on the

set of nonnegative integers. Arnold [4] raised the question as to whether the independence of the random variable $Y_{2,n} - Y_{1,n}$ and the event $\{Y_{1,n} = m\}$ for a fixed $m \geq 1$ implies the distribution F to be geometric (or shifted geometric). In this case, the property is equivalent to a recurrence relation of the type (2.24) subject to a modification that w_n 's are given to be certain functions of v_n 's. In view of the modification involved, we cannot obviously apply our result in Remark 11 directly to identify the solution $\{v_n\}$ to the system of equations in the present case. However, the result of Corollary 4 shows that, under mild conditions assuring certain points to be atoms of F , the independence of $Y_{2,n} - Y_{1,n}$ and $\{Y_{1,n} = m\}$ together with the condition

$$P\{Y_{2,n} - Y_{1,n} > j | Y_{1,n} = m\} \propto P\{Y_{2,n} - Y_{1,n} > j | Y_{1,n} = m + m'\},$$

$$j = 0, 1, \dots, m,$$

for some fixed integer $m' > 0$ characterizes F to be a geometric distribution. This extends a result of Sreehari [31] showing that the independence of $Y_{2,n} - Y_{1,n}$ and $\{Y_{1,n} = m\}$ and the independence of $Y_{2,n} - Y_{1,n}$ and $\{Y_{1,n} = m + m'\}$ for some fixed integer $m' > 0$ characterizes, under some mild conditions, F to be a geometric distribution.

3.2. Krishnaji [12]

In view of the result in Remark 11, it is seen that Theorem 4 of Krishnaji [12] is not correct. This also follows from the counterexample given by Patil and Taillie [18]. The error in Krishnaji's argument appears in the last sentence of the proof in which it is claimed that since $X = A \exp\{A(\theta - 1)\}$ is degenerate, A has to be degenerate. We may, however, point out here that Krishnaji's theorem with the portion " $G(t)$ is non-negative for all real t " in it replaced by " $G(t)$ is infinitely divisible" is valid.

3.3. Shanbhag and Taillie [24]

The following result is an extended version of the Shanbhag-Taillie [29] result and it is a trivial corollary of our result of Corollary 4. (For a proof, based on the Perron-Frobenius theorem, of the Shanbhag-Taillie result, see Alzaid *et al.* [2].)

Let $((a_x, b_x): x = 0, 1, \dots)$ be a sequence vectors with non-negative real components such that $a_x > 0$ for all x and $b_0 > 0$. Let (X, Y) be a random vector with non-negative integer-valued components such that for each x with $P\{X = x\} > 0$, we have

$$P\{Y = y | X = x\} = \frac{a_y b_{x-y}}{c_x}, \quad y = 0, 1, \dots, x,$$

where $\{c_x\}$ is the convolution of $\{a_x\}$ and $\{b_x\}$. Assume that $P\{X-Y=k_0\} > 0$ and $P\{X-Y=k_0+k_1\} > 0$ and denote by γ the largest common divisor of x for which $b_x > 0$. Then the following conditions are equivalent.

- (i) Y and $X-Y$ are independent.
- (ii) $P\{Y=y\} = P\{Y=y|X-Y=k_0\}$, $y=0, 1, \dots$;

$$P\{Y=y|X-Y=k_0\} \propto P\{Y=y|X-Y=k_0+k_1\},$$

$$y=0, 1, \dots, k_0+\gamma-1.$$

- (iii) For some $\theta > 0$ and some periodic sequence $\{q_x: x=0, 1, \dots\}$ with γ as one of its periods

$$P\{X=x\} = q_x c_x \theta^x, \quad x=0, 1, 2, \dots$$

The Shanbhag-Taillie [29] result may be considered to be a variant of Shanbhag's [28] extension of the Rao-Rubin [23] theorem on damage models and is itself an extension of Patil-Taillie [18] result. Patil and Taillie [18] have considered a specialized version of the Shanbhag-Taillie model with $a_x = \pi^x/x!$, $x=0, 1, \dots$, and $b_x = (1-\pi)^x/x!$, $x=0, 1, \dots$, when $\pi \in (0, 1)$ and fixed, and hence with

$$P\{Y=y|X=x\} = \binom{x}{y} \pi^y (1-\pi)^{x-y}, \quad y=0, 1, \dots, x; x \geq 0. \quad (3.1)$$

These latter authors have also shown that if (3.1) is valid together with

$$P\{Y=y\} = P\{Y=y|X-Y=k\}, \quad y=0, 1, \dots; P\{X-Y=k\} > 0 \quad (3.2)$$

for some fixed $k > 0$, then it is not necessary that X be Poisson thus disproving a conjecture of Srivastava and Singh [32]. Our result of the last remark not only shows that the Srivastava-Singh conjecture is false but also identifies under the Shanbhag-Taillie model the class of distributions relative to which the condition (3.2) is valid. In particular, it follows from our result that under the model in question (3.2) is valid with $k=1$, if and only if either $g_x/c_x \propto \theta \lambda_1^{x-1} + (1-\theta) \lambda_2^{x-1}$ for some $\theta < 1$ and $0 < \lambda_1 \leq \lambda_2$ satisfying $\sum_{x=0}^{\infty} b_x \lambda_1^{x-1} = \sum_{x=0}^{\infty} b_x \lambda_2^{x-1}$ or $g_x/c_x \propto \{\theta + (1-\theta)x\} \lambda^x$ for some $\theta \in [0, 1)$ and $\lambda > 0$ satisfying $\sum_{x=0}^{\infty} (x-1) b_x \lambda^x = 0$, where $g_x = P\{X=x\}$.

3.4. Kendall [10, 11], Lindley [16], and Others

The restricted Deny equation (1.2) appears in several places in queueing and storage theories (see, for example, Kendall [10, 11], Lindley [16], and Wishart [33]). In particular, Lindley [16] has shown that the stationary waiting time distribution function corresponding to a $GI/G/1$ queueing system satisfies (1.2) with $H(x) = 0$ for $x < 0$. Indeed our Corollary 2 gives

the expression for H in this case and shows that the distribution in question exists if and only if the relative traffic intensity of the system is less than 1 and that when it exists the distribution is compound geometric; it may, however, be noted here that the results cited are not new and these have appeared in Lindley [16], Feller [9], and elsewhere. The results of our last remark could be applied to obtain certain conclusions of Kendall [11] and Wishart [33] concerning $GI/M/s$ and $GI/E_k/1$ systems, respectively. The result implies that, in either of the two cases, the stationary queue length distribution exists if and only if the corresponding relative traffic intensity is less than 1. Also it yields the known results that in a $GI/M/1$ queueing system the stationary queue length distribution is geometric and in a $GI/M/s$ system the stationary waiting time distribution is exponential but for a discontinuity at zero.

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